Quantum Annealing for Prime Factorization

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Integer Factorization Problem

Classical Factorization

- Methods:
  1. Trial Division($n^{\frac{1}{2}}$): try 2,3,5,...,$n^{\frac{1}{2}}$ to divide n
  2. Shanks, Pollard, Strassen (1970s)($n^{\frac{1}{4}}$): complicated math
  3. CFRAC(1970), QS(1980), NFS(1990) $O(\exp^{\sqrt{(ln(n))(ln(ln(n)))}})$: nonrigorous
  4. Elliptic Curve Method(1985)$O(\exp^{\sqrt{(ln(n))(ln(ln(n)))}})$: probabilistic, nonrigorous
- Limitations:

  EXP time if no randomness, no unproved hypotheses
  Quadratic Sieve Method(subexponential, not polynomial)

Integer Factorization Problem

Quantum Factorization

- Gate Model
  Shor's algo: order-finding
    Exponential speedup: polynomial time
    Hard to physically implement
    Largest Number: 21
- Adiabatic Computation Model
  Convert to optimization problem
  1. NMR
     $f = (N - pq)^2$
  2. Ising Machine: D-wave, ...
     Multiplication Table
  3. Nitrogen-Vacancy (NV) center in diamond
     Multiplication Table

Integer Factorization Problem

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Quantum Adiabatic Computation

Method

1 Define Hamiltonian:

$$\mathcal{H}(t) = (1 - t/T)\mathcal{H}_0 + (t/T)\mathcal{H}_P$$

2 Set system to ground state of initial $\mathcal{H}_0$ (easy)

3* Define final $\mathcal{H}_P$ s.t. its ground state encodes problem's solution

4 System slowly evolves to final state(eigenvector w.r.t. smallest eigenvalue),measure it to get soln

How to encode H_p

$$f_p(s_1, s_2, ..., s_n) = \sum_{k=1}^{K} \sum_{j_1,...,j_k=1}^{n} J_{j_1...j_k} s_{j_1} ... s_{j_k}, s_i = \pm 1$$

is the energy func.(eigenvalue func.) of Hamiltonian

$$\mathcal{H}_p(\sigma_z^{(1)}, \sigma_z^{(2)}, ..., \sigma_z^{(n)}) = \sum_{k=1}^{K} \sum_{j_1,...,j_k=1}^{n} J_{j_1...j_k} \sigma_z^{(j_1)} ... \sigma_z^{(j_k)}$$

Here $\sigma_z^{(i)} = \overbrace{I \otimes I \otimes ... \otimes I}^{i-1} \otimes \sigma_z \otimes \overbrace{I \otimes ... \otimes I}^{n-i}$ with eigenvector $|x_1 x_2 ... x_n\rangle, x_i = \{0, 1\}, s_i = (-1)^{x_i}$

Ising model

Definition

$$\mathcal{H}_p(\sigma_z^{(1)}, \sigma_z^{(2)}, ..., \sigma_z^{(n)}) = \sum_{i=1}^{n} h_i \sigma_z^{(i)} + \sum_{i,j=1}^{n} J_{ij} \sigma_z^{(i)} \sigma_z^{(j)}$$

h_i: external magnetic

J_{ij}: interaction between two adjacent sites i, j

Ising Model $\leftrightarrow$ Quadratic Function

Our Method for Factorization[1]

- Define Cost Function
  - Direct Method: $f = (N - pq)^2$
  - Modified Multiplication Table Method
- Order Reducing Method
  - $x, y, z \in \{0, 1\}$
    $xy = z$ iff $xy - 2xz - 2yz + 3z = 0$,
    $xy \neq z$ iff $xy - 2xz - 2yz + 3z > 0$
    $\Rightarrow x_1, x_2, x_3 \in \{0, 1\}$
    $\min(x_1x_2x_3) = \min_{x_4=x_1x_2}(x_4x_3 + 2(x_1x_2 - 2x_1x_4 - 2x_2x_4 + 3x_4))$
    $\min(-x_1x_2x_3) = \min_{x_4=x_1x_2}(-x_4x_3 + 2(x_1x_2 - 2x_1x_4 - 2x_2x_4 + 3x_4))$

[1] Shuxian Jiang et al. "Quantum Annealing for Prime Factorization". In: *arXiv preprint arXiv:1804.02733* (2018).

Cost Function: Direct Method

For $N = pq$

$p = (p_{l-1}p_{l-2}...p_1 1)_2, q = (q_{l'-1}q_{l'-2}...q_1 1)_2$

$$
\begin{aligned}
\text{Define} \quad f &= (N - pq)^2 \\
&= [N - (\sum_{i=1}^{l-1} 2^i p_i + 1)(\sum_{j=1}^{l'-1} 2^j q_j + 1)]^2 \\
&= N^2 + (\sum_{i=1}^{l-1} 2^i p_i + 1)^2 (\sum_{j=1}^{l'-1} 2^j q_j + 1)^2 \\
&\quad -2N(\sum_{i=1}^{l-1} 2^i p_i + 1)(\sum_{j=1}^{l'-1} 2^j q_j + 1)
\end{aligned}
$$

Reduce Order:

$$\min(x_1x_2x_3) = \min_{x_4 = x_1x_2}(x_4x_3 + 2(x_1x_2 - 2x_1x_4 - 2x_2x_4 + 3x_4))$$

Cost Function: Direct Method

For $N = 15 = 5 \times 3$

$p = (x_1 1)_2 = x_1 * 2 + 1, q = (x_2 x_3 1)_2 = x_2 * 2^2 + x_3 * 2 + 1$

$$
\begin{aligned}
& f(x_1, x_2, x_3) \\
= \; & (N - pq)^2 \\
= \; & [15 - (x_1 * 2 + 1)(x_3 * 2^2 + x_2 * 2 + 1)]^2 \\
= \; & 15^2 + (2x_1 + 1)^2(4x_3 + 2x_2 + 1)^2 - 30(2x_1 + 1)(4x_3 + 2x_2 + 1) \\
= \; & 128x_1x_2x_3 - 56x_1x_2 - 48x_1x_3 + 16x_2x_3 - 52x_1 - 52x_2 - 96x_3 \\
& +196
\end{aligned}
$$

Reduce Order:

$$
\begin{aligned}
& f'(x_1, x_2, x_3, x_4) \\
= \; & 128(x_4x_3 + 2(x_1x_2 - 2x_1x_4 - 2x_2x_4 + 3x_4)) - 56x_1x_2 \\
& -48x_1x_3 + 16x_2x_3 - 52x_1 - 52x_2 - 96x_3 + 196
\end{aligned}
$$

Cost Function: Modified Multiplication Table Method

Table: Multiplication table for $11 \times 13 = 143$ in binary.

	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
p					1	p_2	p_1	1
q					1	q_2	q_1	1
					1	p_2	p_1	1
				q_1	$p_2 q_1$	$p_1 q_1$	q_1	
			q_2	$p_2 q_2$	$p_1 q_2$	q_2		
		1	p_2	p_1	1			
carries		c_4	c_3	c_2	c_1			
$p \times q = 143$	1	0	0	0	1	1	1	1

$$2(p_2 + p_1 q_1 + q_2) + (p_1 + q_1) \quad = \quad 8c_2 + 4c_1 + 3$$

...

Cost Function: Modified Multiplication Table Methods

From

$$2(p_2 + p_1 q_1 + q_2) + (p_1 + q_1) \;\; = \;\; 8c_2 + 4c_1 + 3$$

We got

$$f_1 = (2p_2 + 2p_1 q_1 + 2q_2 - 8c_2 - 4c_1 + p_1 + q_1 - 3)^2$$

Reduce order

$$\min(\pm x_1 x_2 x_3) \;\; = \;\; \min_{x_4 = x_1 x_2}(\pm x_4 x_3 + 2(x_1 x_2 - 2x_1 x_4 - 2x_2 x_4 + 3x_4))$$

Embed to D-wave

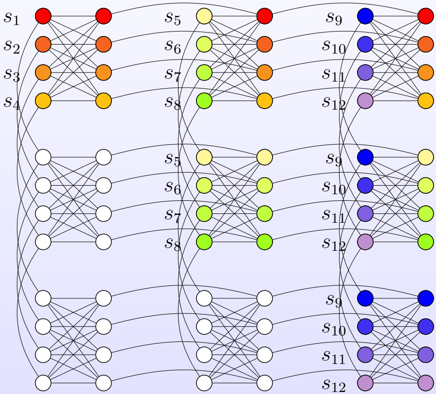
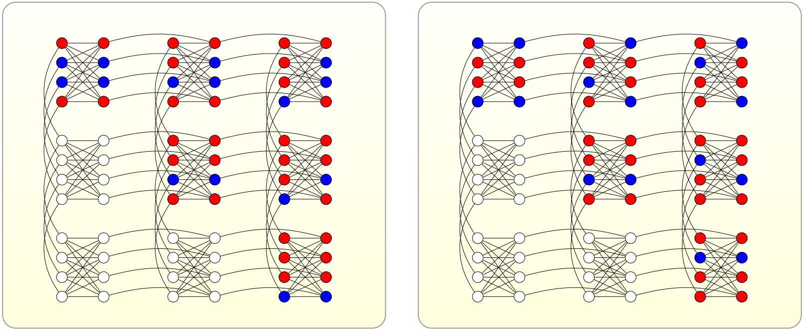


Figure: Embed problem graph of factoring 143 using method one to Chimera graph

Embed to D-wave



(a) Result 13×11 (b) Result 11×13

Figure: Final ground state of factoring 143. Red nodes:+1, Blue nodes:-1. (a) $s_1 = 1, s_2 = -1, s_3 = -1, s_4 = 1$ ($p = 1101, q = 1011$). (b) $s_1 = -1, s_2 = 1, s_3 = 1, s_4 = -1$ ($p = 1011, q = 1101$).

Cost Function: Modified Multiplication Table Method

Table: Multiplication table for $659 \times 571 = 376289$ in binary.

2^{18}	2^{17}	2^{16}	2^{15}	2^{14}	2^{13}	2^{12}	2^{11}	2^{10}	2^{9}	2^{8}	2^{7}	2^{6}	2^{5}	2^{4}	2^{3}	2^{2}	2^{1}	2^{0}
									1	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	1
									1	q_8	q_7	q_6	q_5	q_4	q_3	q_2	q_1	1
									1	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	1
								q_1	p_8q_1	p_7q_1	p_6q_1	p_5q_1	p_4q_1	p_3q_1	p_2q_1	p_1q_1	q_1	
							q_2	p_8q_2	p_7q_2	p_6q_2	p_5q_2	p_4q_2	p_3q_2	p_2q_2	p_1q_2	q_2		
						q_3	p_8q_3	p_7q_3	p_6q_3	p_5q_3	p_4q_3	p_3q_3	p_2q_3	p_1q_3	q_3			
					q_4	p_8q_4	p_7q_4	p_6q_4	p_5q_4	p_4q_4	p_3q_4	p_2q_4	p_1q_4	q_4				
				q_5	p_8q_5	p_7q_5	p_6q_5	p_5q_5	p_4q_5	p_3q_5	p_2q_5	p_1q_5	q_5					
			q_6	p_8q_6	p_7q_6	p_6q_6	p_5q_6	p_4q_6	p_3q_6	p_2q_6	p_1q_6	q_6						
		q_7	p_8q_7	p_7q_7	p_6q_7	p_5q_7	p_4q_7	p_3q_7	p_2q_7	p_1q_7	q_7							
	q_8	p_8q_8	p_7q_8	p_6q_8	p_5q_8	p_4q_8	p_3q_8	p_2q_8	p_1q_8	q_8								
1	p_8	p_7	p_6	p_5	p_4	p_3	p_2	p_1	1									
c_{14}	c_{13}			c_9	c_8	c_7	c_6	c_5	c_4	c_3		c_2	c_1					
		c_{12}	c_{11}	c_{10}														
1	0	1	1	0	1	1	1	1	0	1	1	1	1	0	0	0	0	1

Experimental Results

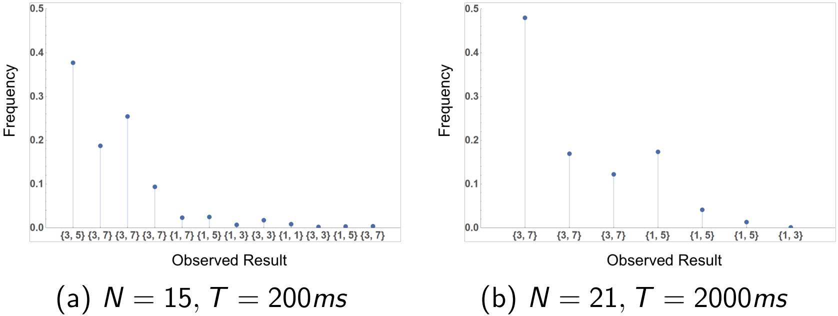


Figure: D-wave Results for Method One: rates of getting different solutions.

Experimental Results

Table: D-wave Results for Method Two

N	factors	logic#qb	phy#qb	embed	sol
35	7× 5	5	8	✓	✓
143	13× 11	12	48	✓	✓
391	23× 17	20	113	✓	✓
1517	41× 37	30	270	✓	✓
8137	103 × 79	43	535	✓	✓
56153	241× 233	58	1085	✓	–
249919	509×491	77	1803	✓	–
376289	659×571	94	–	–	–