Markowitz Portfolio Optimization with a Quantum Annealer

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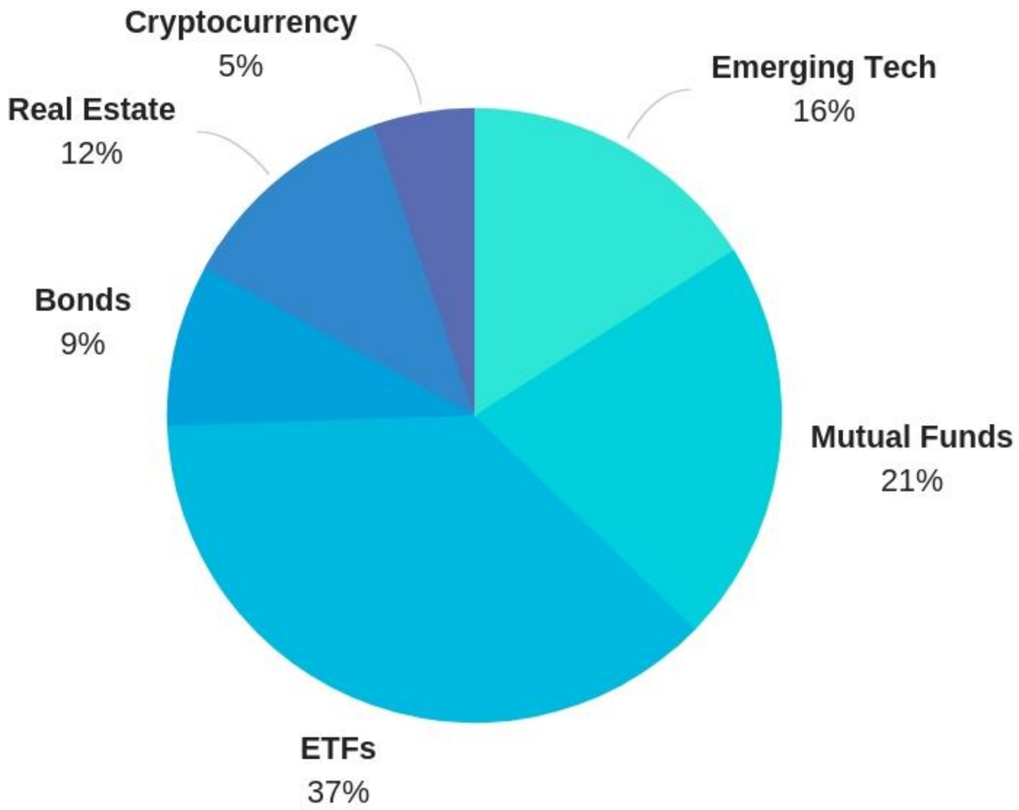
Markowitz Portfolio Selection



Goals:

- Maximize returns
- Minimize risk
- Stay within budget

Inputs:

- Historical price data
- Budget
- Risk tolerance

Output:

- Binary list of investments: the portfolio

Budget: $200 **Risk Tolerance: Low**

Investment	Price	Expected Return	1= buy 0 = pass
Apple	$223	- $0.20	0
FitBit	$5.90	+ $0.10	1

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Application to Cryptocurrency

Many investors are including cryptocurrency as part of their portfolios.

- A digital currency designed as a medium of exchange
- Uses cryptography and blockchain to verify and secure each transaction
- Many treat it as an investment
- Volatile: **pro** → potential for high returns

  con → potential for significant losses

Application to Cryptocurrency

- Cryptocurrencies can be divided into any desired fraction based on the budget.
- Normalize purchase price to the budget.
- Use Binary Fractional series:

$$\text{fractions} = \frac{1}{2^0}, \frac{1}{2^1}, \frac{1}{2^2}, \cdots, \frac{1}{2^n}$$

Bitcoin

Budget = $200

	100%	50%	25%	12.5% ...
06-16-2018	97.44	48.72	24.36	12.18
06-17-2018	107.40	53.50	26.75	13.37
06-18-2018	94.71	47.36	23.67	11.84
06-19-2018	101.79	50.89	25.45	12.73
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Today	200	100	50	25

Special thanks to Benjamin Stump
for the binary fractional series idea

Markowitz Formulation

$$\min_{x} f(x)$$

$$f(x) = -\theta_1 \sum_i x_i r_{ii} x_i \ + \theta_2 \sum_i (x_i p_i x_i - b)^2 + \theta_3 \sum_{i,j} x_i cov(p_i, p_j) x_j$$

$$s.t.$$
$$\theta_1 + \theta_2 + \theta_3 = 1$$

$$\boldsymbol{x_i = 1 \ \rightarrow buy \qquad x_i = 0 \ \rightarrow don't\ buy}$$

- $b = budget, \ p_i = asset\ price, r_i = asset's\ expected\ return \ , \quad x_i \in \{0, 1\}$
- Weights: $\theta_1 = expected\ returns, \theta_2 = budget\ constraint, \ \theta_3 = diversification$

Markowitz Formulation

$$\min_x f(x)$$

Expected Returns

$$f(x) = -\theta_1 \sum_i x_i r_{ii} x_i + \theta_2 \sum_i (x_i p_i x_i - b)^2 + \theta_3 \sum_{i,j} x_i cov(p_i, p_j) x_j$$

$$s.t.$$
$$\theta_1 + \theta_2 + \theta_3 = 1$$

$$\boldsymbol{x_i = 1 \ \rightarrow buy \qquad x_i = 0 \ \rightarrow don't\, buy}$$

- $b = budget,\ p_i = asset\ price, r_i = asset's\ expected\ return\ ,\quad x_i \epsilon \{0, 1\}$
- Weights: $\theta_1 = expected\ returns, \theta_2 = budget\ constraint,\ \theta_3 = diversification$

Markowitz Formulation

$$\min_{x} f(x)$$

Budget Penalty

$$f(x) = -\theta_1 \sum_i x_i r_{ii} x_i + \theta_2 \sum_i (x_i p_i x_i - b)^2 + \theta_3 \sum_{i,j} x_i cov(p_i, p_j) x_j$$

$$s.t.$$
$$\theta_1 + \theta_2 + \theta_3 = 1$$

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Markowitz Formulation

$$\min_{x} f(x)$$

Diversification

$$f(x) = -\theta_1 \sum_i x_i r_{ii} x_i + \theta_2 \sum_i (x_i p_i x_i - b)^2 + \theta_3 \sum_{i,j} x_i cov(p_i, p_j) x_j$$

$$s.t.$$
$$\theta_1 + \theta_2 + \theta_3 = 1$$

$$\boldsymbol{x_i = 1 \rightarrow buy \qquad x_i = 0 \rightarrow don't\ buy}$$

- $b = budget,\ p_i = asset\ price, r_i = asset's\ expected\ return$, $x_i \epsilon \{0, 1\}$
- Weights: $\theta_1 = expected\ returns, \theta_2 = budget\ constraint,\ \theta_3 = diversification$

QUBO to Quantum Ising

QUBO: $x \epsilon \{0, 1\}$

$$f(x) = -\theta_1 \sum_i x_i r_{ii} x_i \ + \theta_2 \sum_i (x_i p_i x_i - b)^2 + \theta_3 \sum_{i,j} x_i cov(p_i, p_j) x_j$$

$$f(x) = Q_{i,j}^{\sim} x_i x_j + q_i x_i$$

$$q_i = Q_{ii} \quad and \quad Q_{i,j}^{\sim} = Q_{i,j} \ (i \neq j)$$

Ising: $y \epsilon \{-1, 1\}$

$$J_{i,j} = \frac{1}{4} Q_{i,j}^{\sim} \quad h_i = \frac{q_i}{2} + \sum_j J_{i,j}$$

$$\gamma = \frac{1}{4} \Sigma_{i,j} Q_{i,j} + \frac{1}{2} \Sigma_i q_i$$

$$\min_x f(x)$$

$$f(x) = \sum_i h_i y_i + \sum_{i,j} J_{i,j} y_i y_j + \gamma$$

$$\hat{H} = -\sum_{i,j} J_{i,j} \hat{z}_i \hat{z}_j - \sum_i h_i \hat{z}_i + \gamma$$

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Coupler Strengths

$$J_{i,j} = \frac{1}{4} Q_{i,j}^{\sim} \qquad h_i = \frac{q_i}{2} + \sum_j J_{i,j}$$

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$$\min_x f(x)$$

$$f(x) = \sum_i h_i y_i + \sum_{i,j} J_{i,j} y_i y_j + \gamma$$

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$$f(x) = Q_{\widetilde{i,j}} x_i x_j + q_i x_i$$

$$q_i = Q_{ii} \quad and \quad Q_{\widetilde{i,j}} = Q_{i,j} \ (i \neq j)$$

Ising: $y \epsilon \{-1,1\}$

Qubit Weights

$$J_{i,j} = \frac{1}{4} Q_{\widetilde{i,j}} \qquad h_i = \frac{q_i}{2} + \sum_j J_{i,j}$$

$$\gamma = \frac{1}{4} \sum_{i,j} Q_{i,j} + \frac{1}{2} \sum_i q_i$$

$$\min_x f(x)$$

$$f(x) = \sum_i h_i y_i + \sum_{i,j} J_{i,j} y_i y_j + \gamma$$

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QUBO to Quantum Ising

QUBO: $x \epsilon \{0, 1\}$

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$$\gamma = \frac{1}{4} \Sigma_{i,j} Q_{i,j} + \frac{1}{2} \Sigma_i q_i$$

Constant

$$\min_x f(x)$$

$$f(x) = \sum_i h_i y_i + \sum_{i,j} J_{i,j} y_i y_j + \gamma$$

$$\hat{H} = -\sum_{i,j} J_{i,j} \hat{z}_i \hat{z}_j - \sum_i h_i \hat{z}_i + \gamma$$

QUBO to Quantum Ising

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Ising Form

$$\min_x f(x)$$

$$f(x) = \sum_i h_i y_i + \sum_{i,j} J_{i,j}\, y_i y_j + \gamma$$

$$\hat{H} = -\sum_{i,j} J_{i,j} \hat{z}_i \hat{z}_j - \sum_i h_i \hat{z}_i + \gamma$$

QUBO to Quantum Ising

QUBO: $x \in \{0,1\}$

$$f(x) = -\theta_1 \sum_i x_i r_{ii} x_i + \theta_2 \sum_i (x_i p_i x_i - b)^2 + \theta_3 \sum_{i,j} x_i cov(p_i, p_j) x_j$$

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$$\min_x f(x)$$

$$f(x) = \sum_i h_i y_i + \sum_{i,j} J_{i,j} y_i y_j + \gamma$$

Quantum Ising

$$\hat{H} = -\sum_{i,j} J_{i,j} \hat{z}_i \hat{z}_j - \sum_i h_i \hat{z}_i + \gamma$$

Solving on D-Wave 2000Q

D-Wave 2000Q:

- 2048 qubits
- Max job length of $3s$
- Anneal time: $5\mu s$, $100\mu s$, and $250\mu s$

Parameters:

- $\theta_1, \theta_2, \theta_3$
- Number of assets
- Historical Price Data

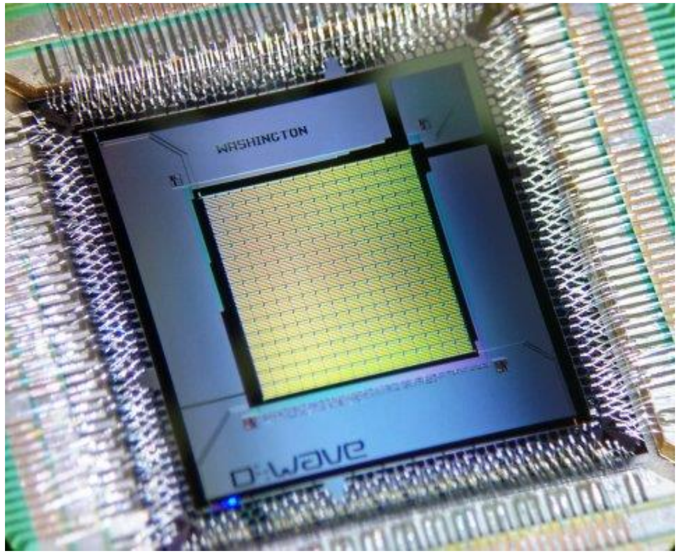
Outputs:

- Portfolio:

[-1, 1, 1, 1, -1, -1…] →

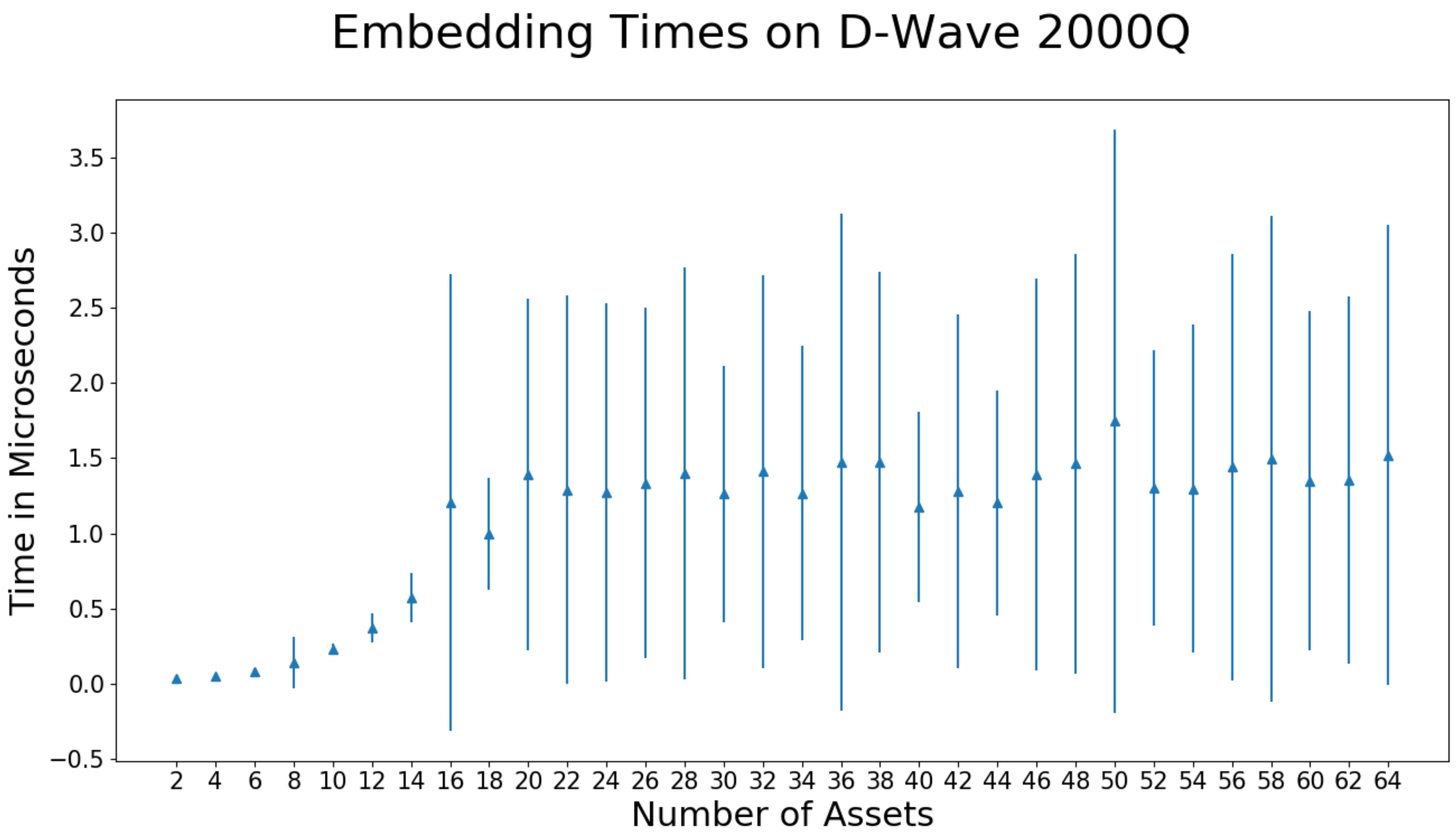
[0, 1, 1, 1, 0, 0…]

- Portfolio Value: the cost of the portfolio



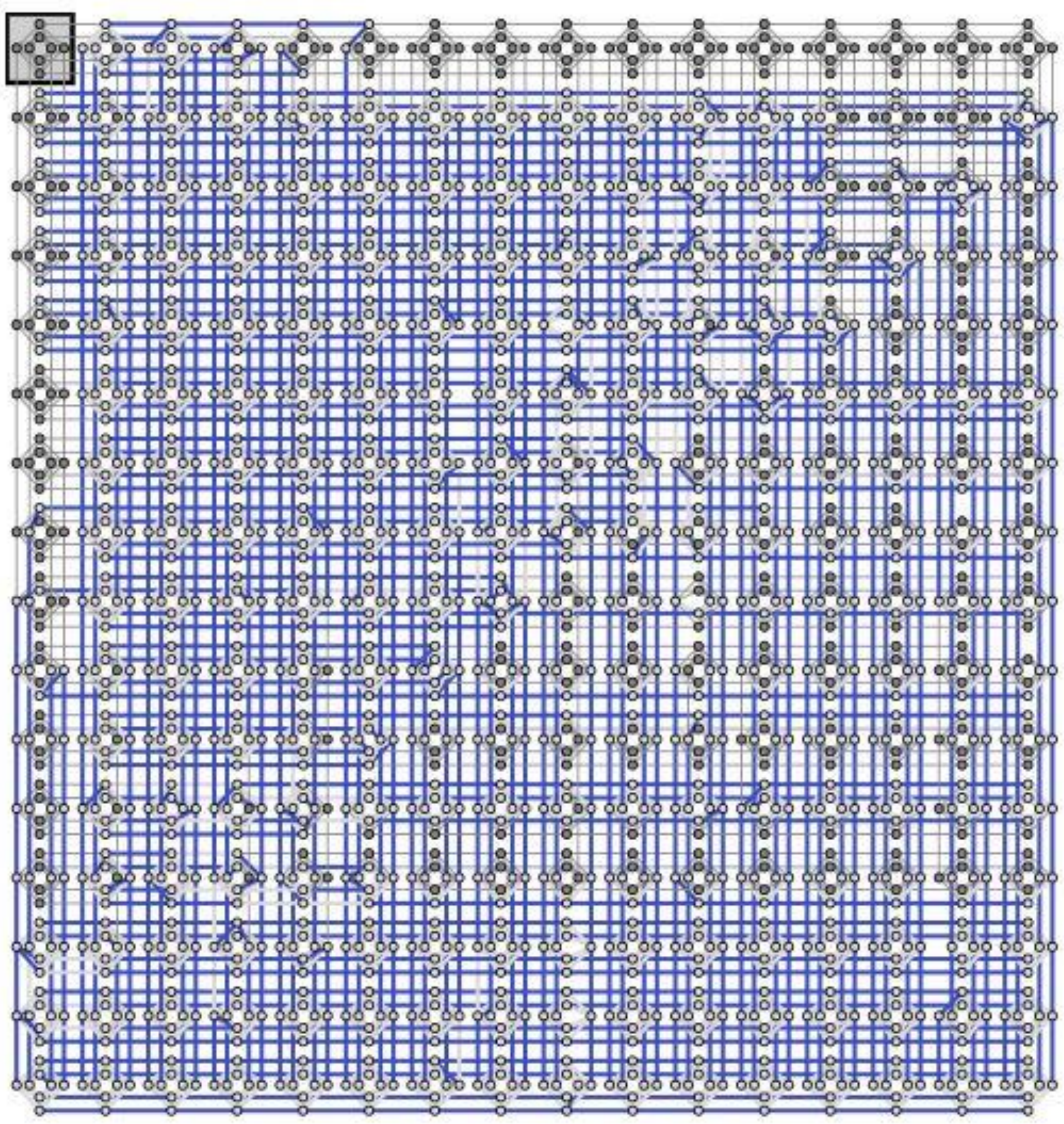
Embedding

- **Embedding Time:**
  - Average over 100 problems
  - Up to 64 assets



Embedding

- Fully connected graph, 65 assets



Problem Setup

Generate price data:

- 100 points in time $\{p_1, p_2, \ldots \ldots p_{100}\}$
- 1,000 problems
- 20 assets
- Random Variability of $\pm 25\%$ between historical price points

	Crypto 1	Crypto 2	Crypto 3	Crypto 4	Crypto 5
Day 1:	**140.4786**	**231.4524**	**847.4452**	**276.9872**	**785.5279**
Day 2:	**166.8358**	**277.8494**	**565.0754**	**188.9914**	**723.1995**
Day 3:	**161.4003**	**346.2087**	**885.931**	**222.8781**	**769.9646**
Day4:	**121.0639**	**216.6541**	**837.3057**	**239.5639**	**763.3887**
Day5:	**120.9109**	**350.5204**	**858.1819**	**259.3726**	**918.4838**
Day 6:	**147.8595**	**265.5617**	**721.3764**	**258.4228**	**1042.285**
Day 7:	**146.3595**	**240.6285**	**558.813**	**173.5751**	**887.7815**

D-Wave:

- Number of runs = 10,000

Brute Force Solver:

- Searches exact solution (global minimum)

Problem Setup

Generate price data:

- 100 points in time $\{p_1, p_2, \ldots \ldots p_{100}\}$
- 1,000 problems
- 20 assets
- Random Variability of $\pm 25\%$ between historical price points

Budget: $200

	Crypto 1	Crypto 2	Crypto 3	Crypto 4	Crypto 5
Day 1:	140.4786	231.4524	847.4452	276.9872	785.5279
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Day 6:	147.8595	265.5617	721.3764	258.4228	1042.285
Price:	146.3595	240.6285	558.813	173.5751	887.7815

Set last day to purchasing price

D-Wave:

- Number of runs = 10,000

Brute Force Solver:

- Searches exact solution (global minimum)

Problem Setup

Generate price data:

- 100 points in time $\{p_1, p_2, \dots \dots p_{100}\}$
- 1,000 problems
- 20 assets
- Random Variability of $\pm 25\%$ between historical price points

Budget: $200

	Crypto 1	Crypto 2	Crypto 3	Crypto 4	Crypto 5
Day 1:	194.1191	190.8239	488.6322	303.4121	97.7464
Day 2:	220.4763	237.2209	206.2624	215.4163	35.418
Day 3:	215.0408	305.5802	527.118	249.303	82.1831
Day4:	174.7044	176.0256	478.4927	265.9888	75.6072
Day5:	174.5514	309.8919	499.3689	285.7975	230.7023
Day 6:	201.5	224.9332	362.5634	284.8477	354.5035
Price:	200	200	200	200	200

Normalize prices to budget

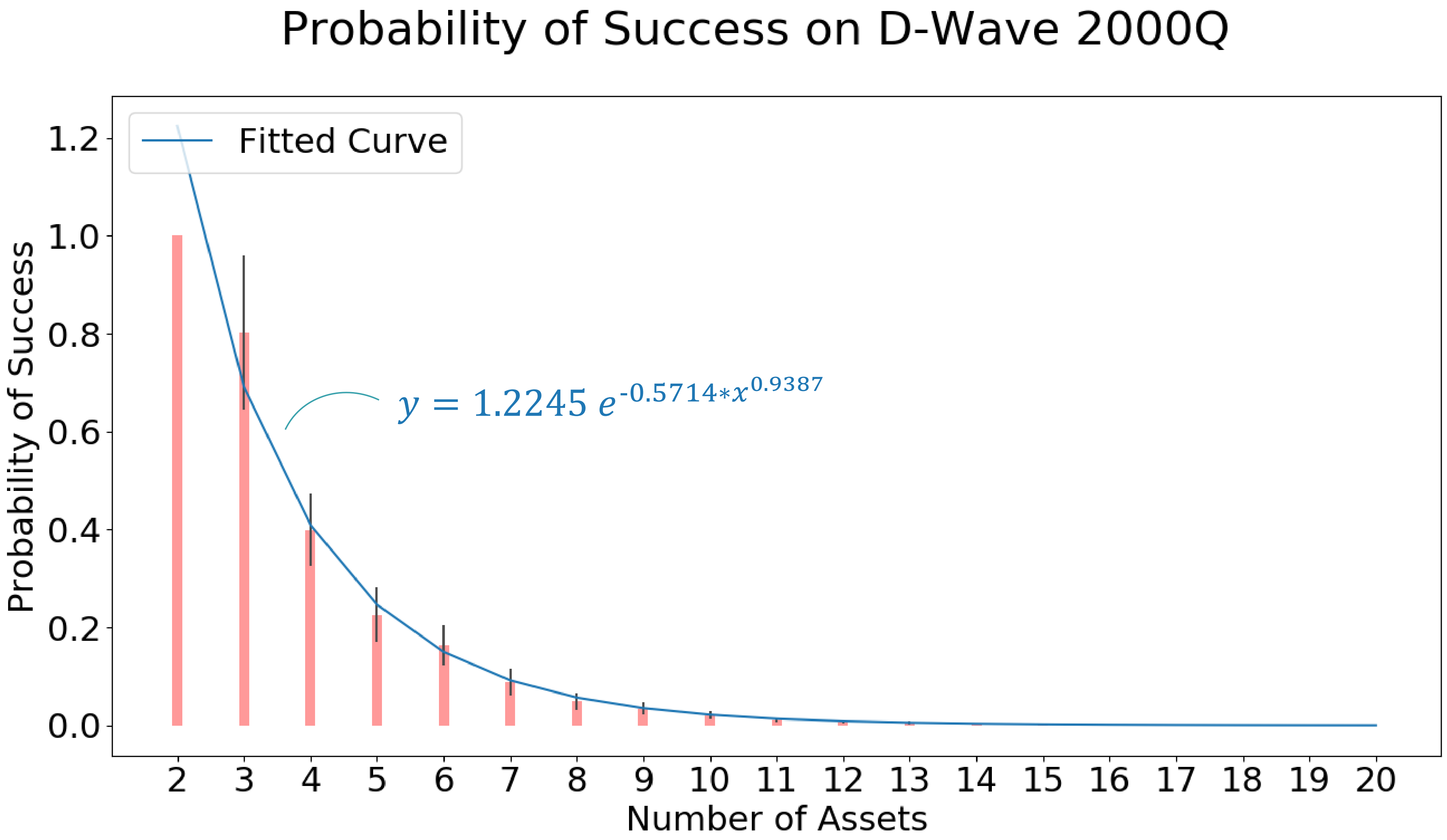
D-Wave:

- Number of runs = 10,000

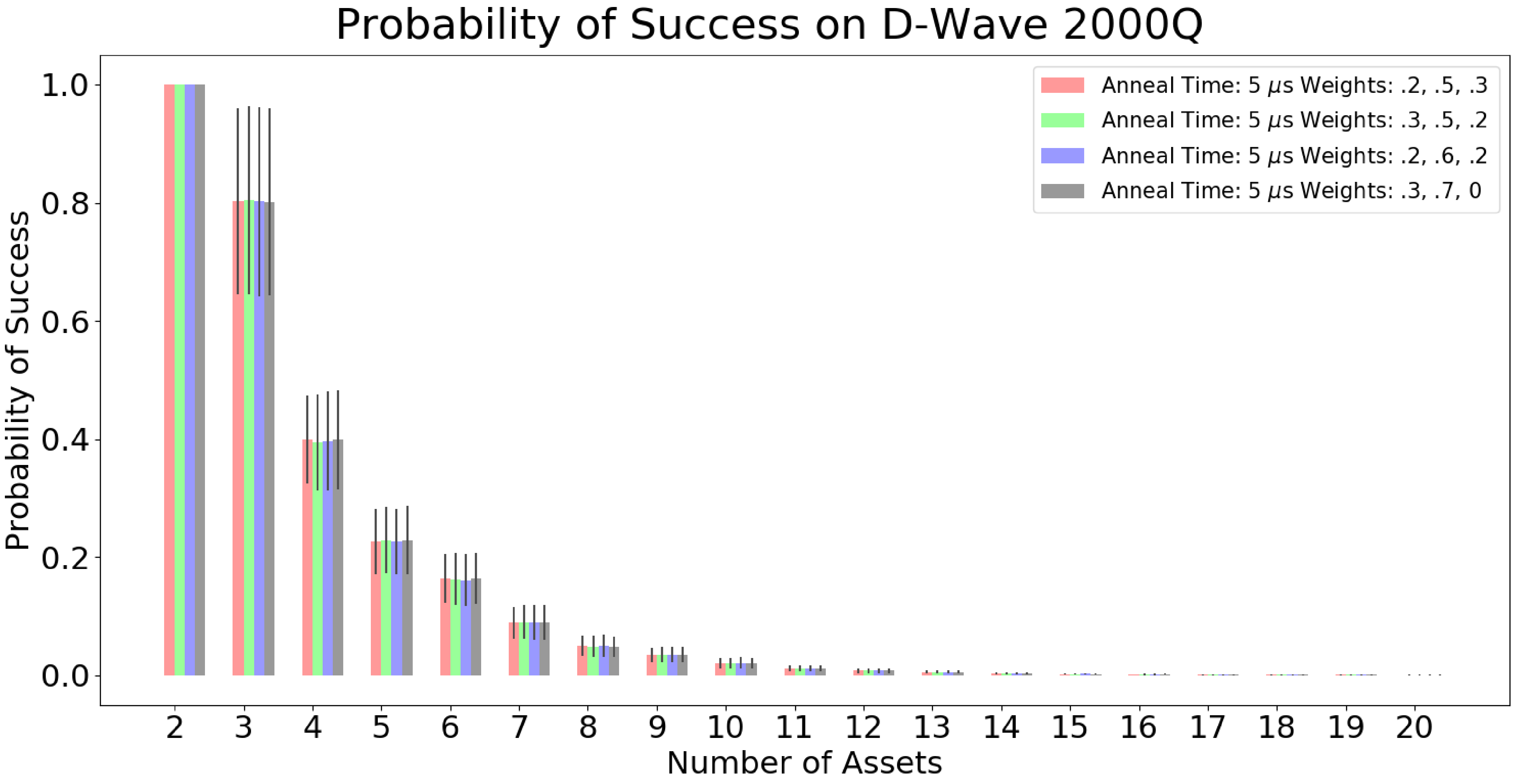
Brute Force Solver:

- Searches exact solution (global minimum)

Best Fit for Probability of Success



Probability of Success: Variation in Weights



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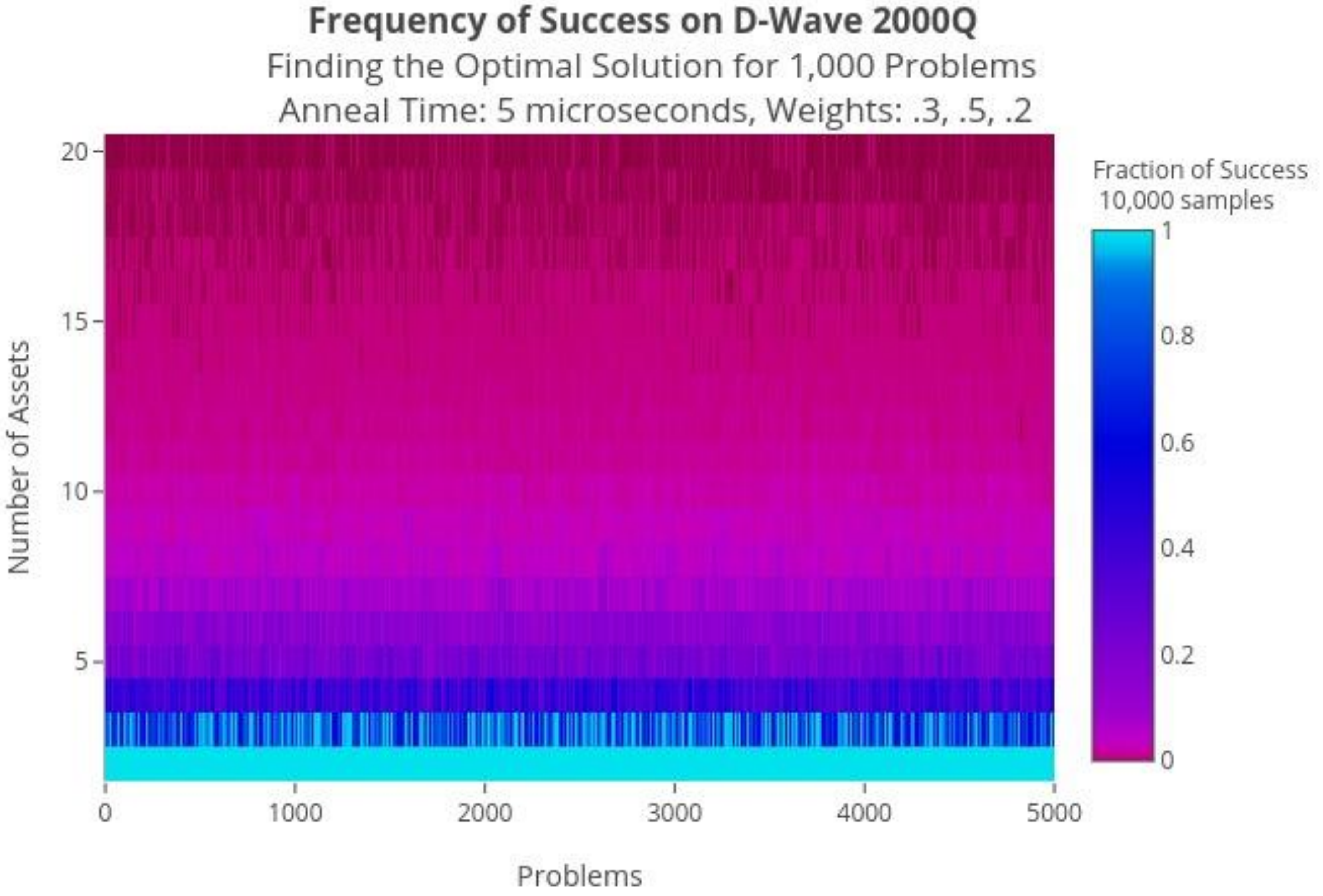
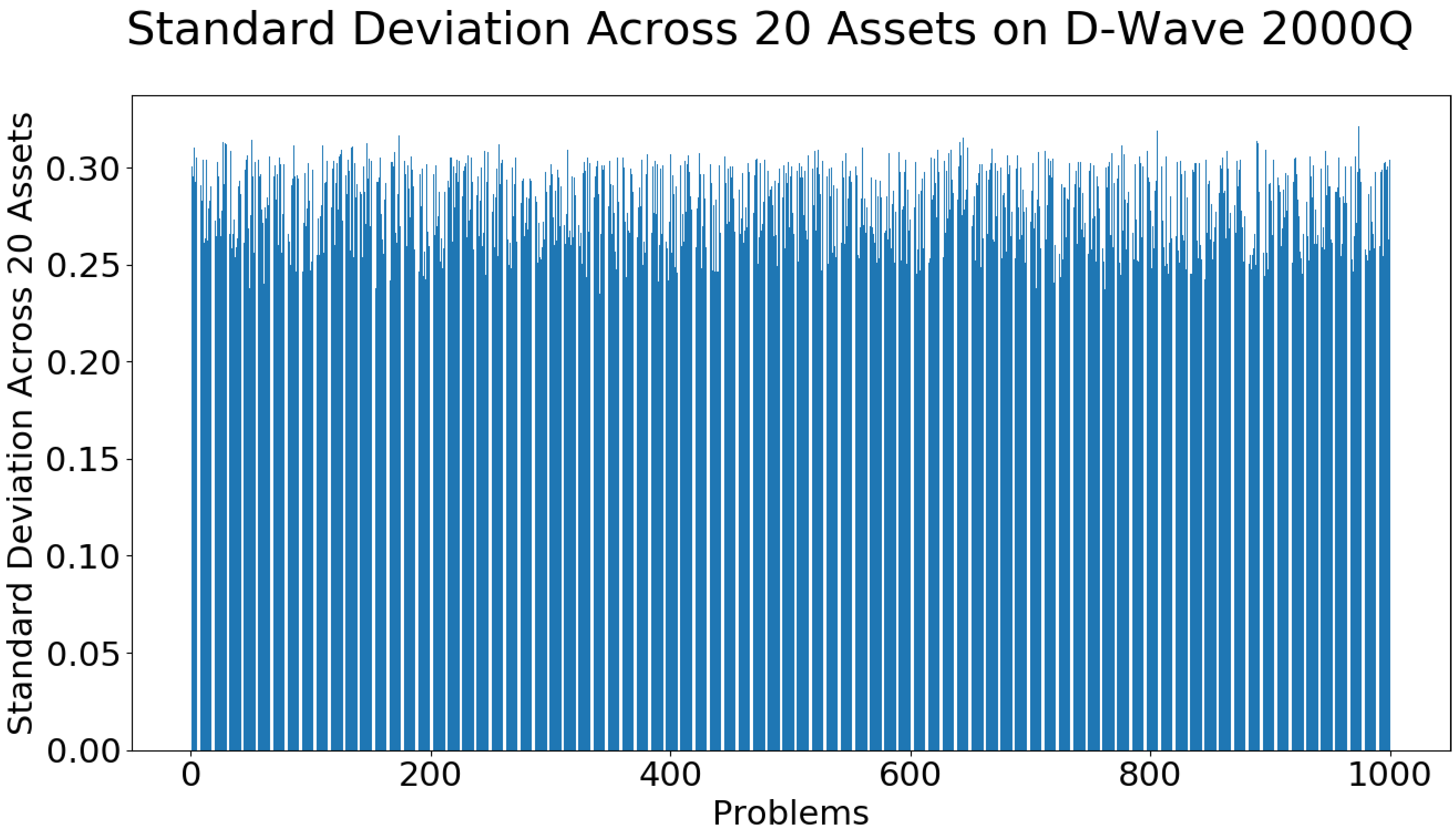
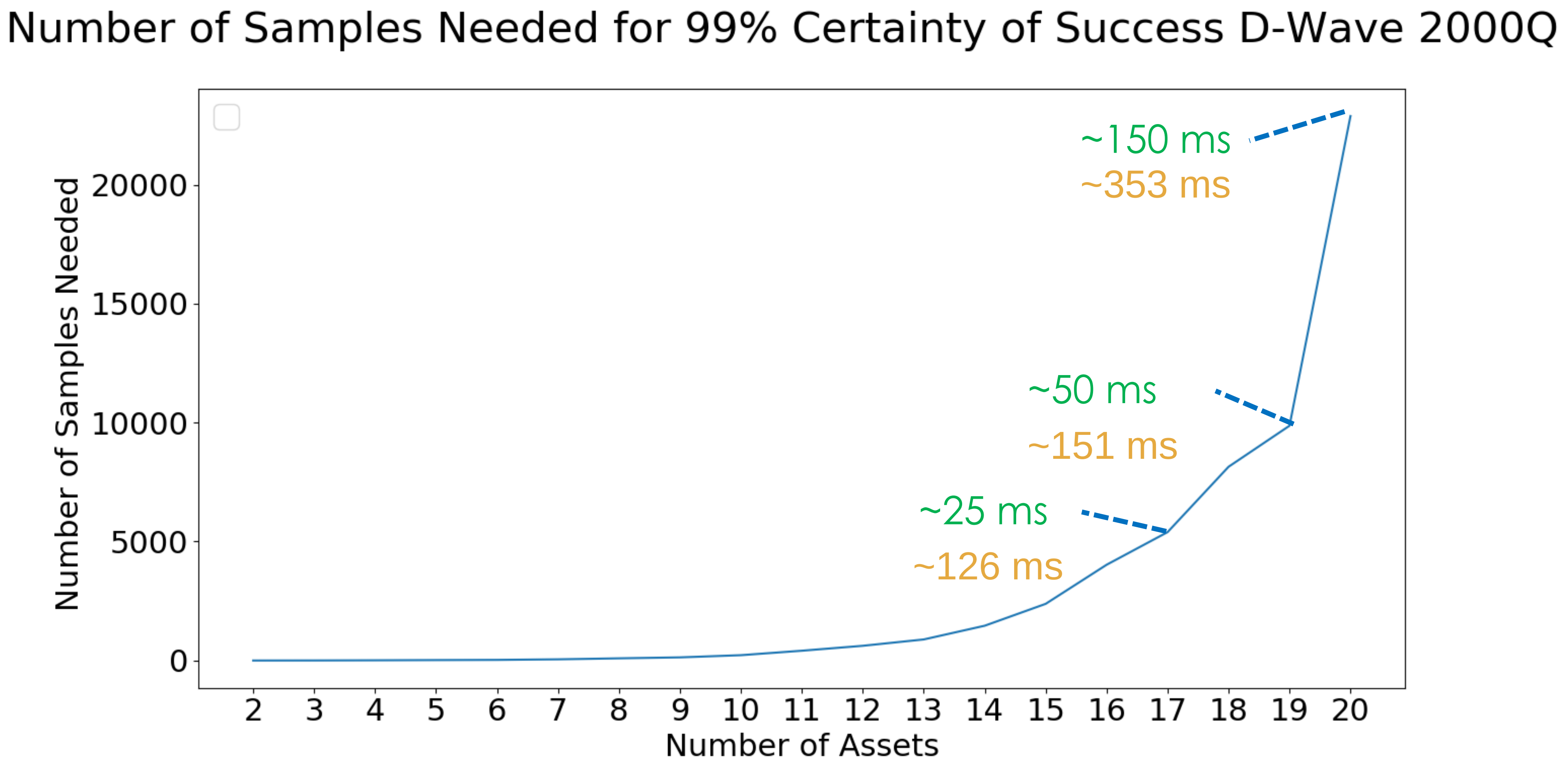
Probability of Success



Standard Deviation Across Problems



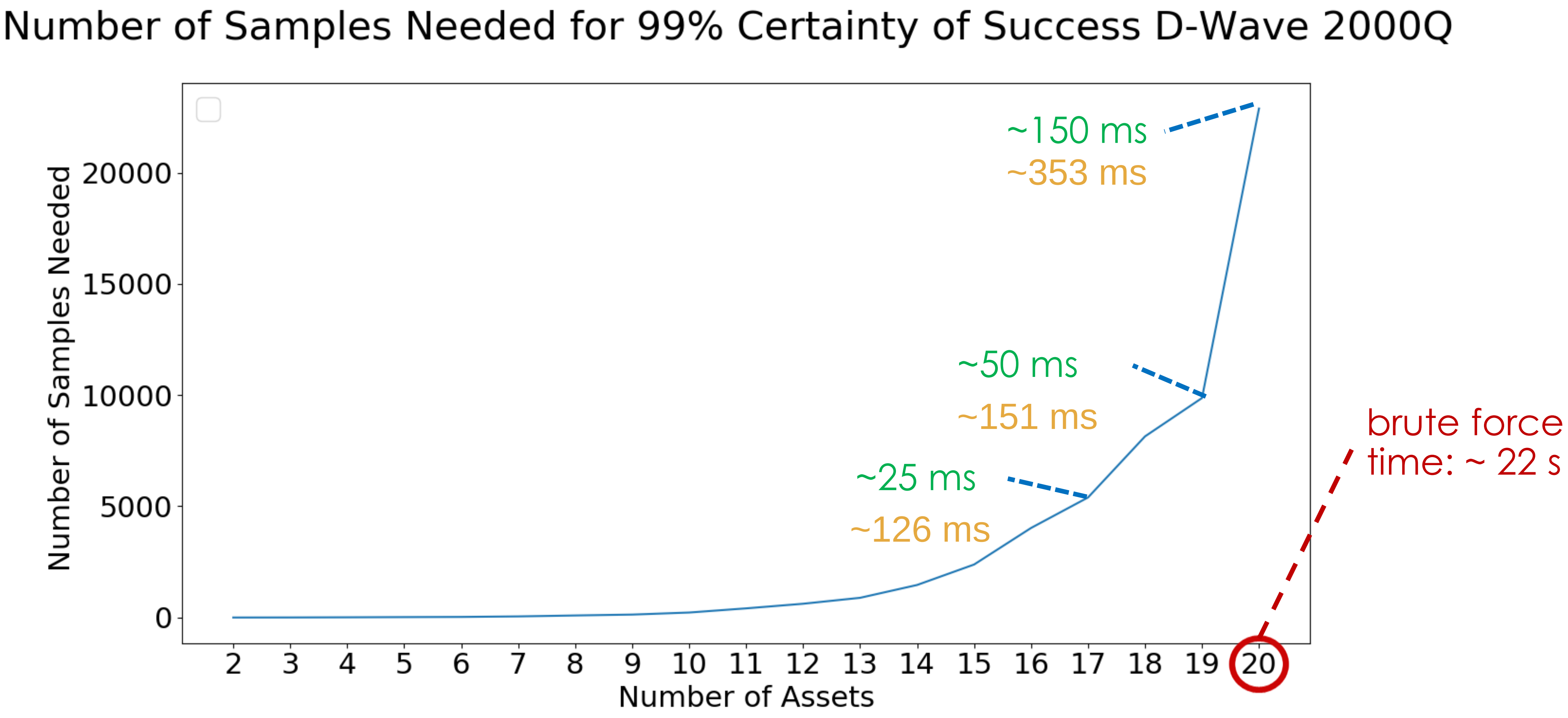
Number of Samples



$n \geq \frac{\log(1-p_a)}{\log(1-p_s)}$ where n is the number of samples, p_a is the desired accuracy, and p_s is probability of success

Total D-Wave time is calculated from anneal time + access time + post-processing overhead for each 10,000 sample job.

Number of Samples



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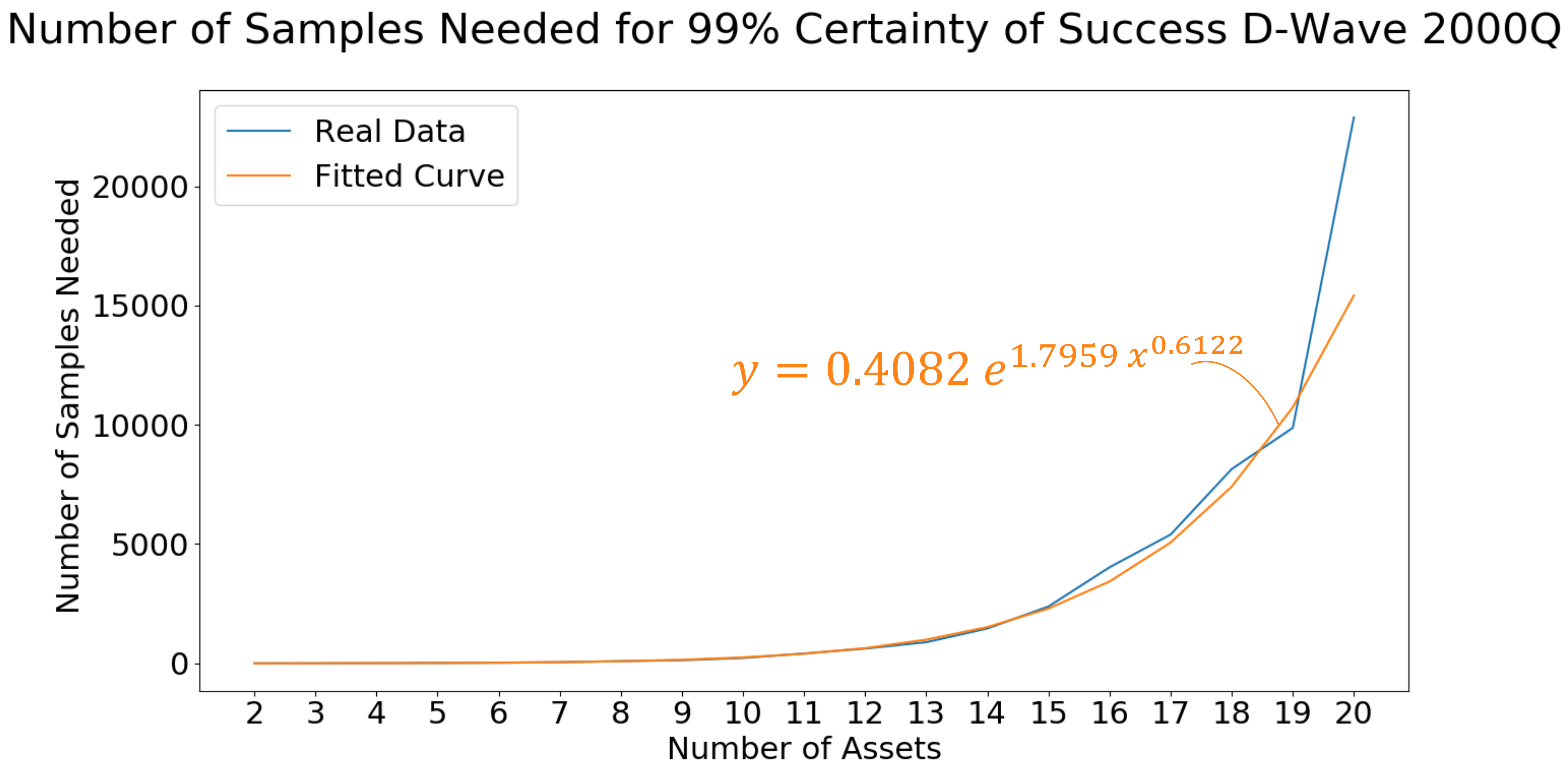
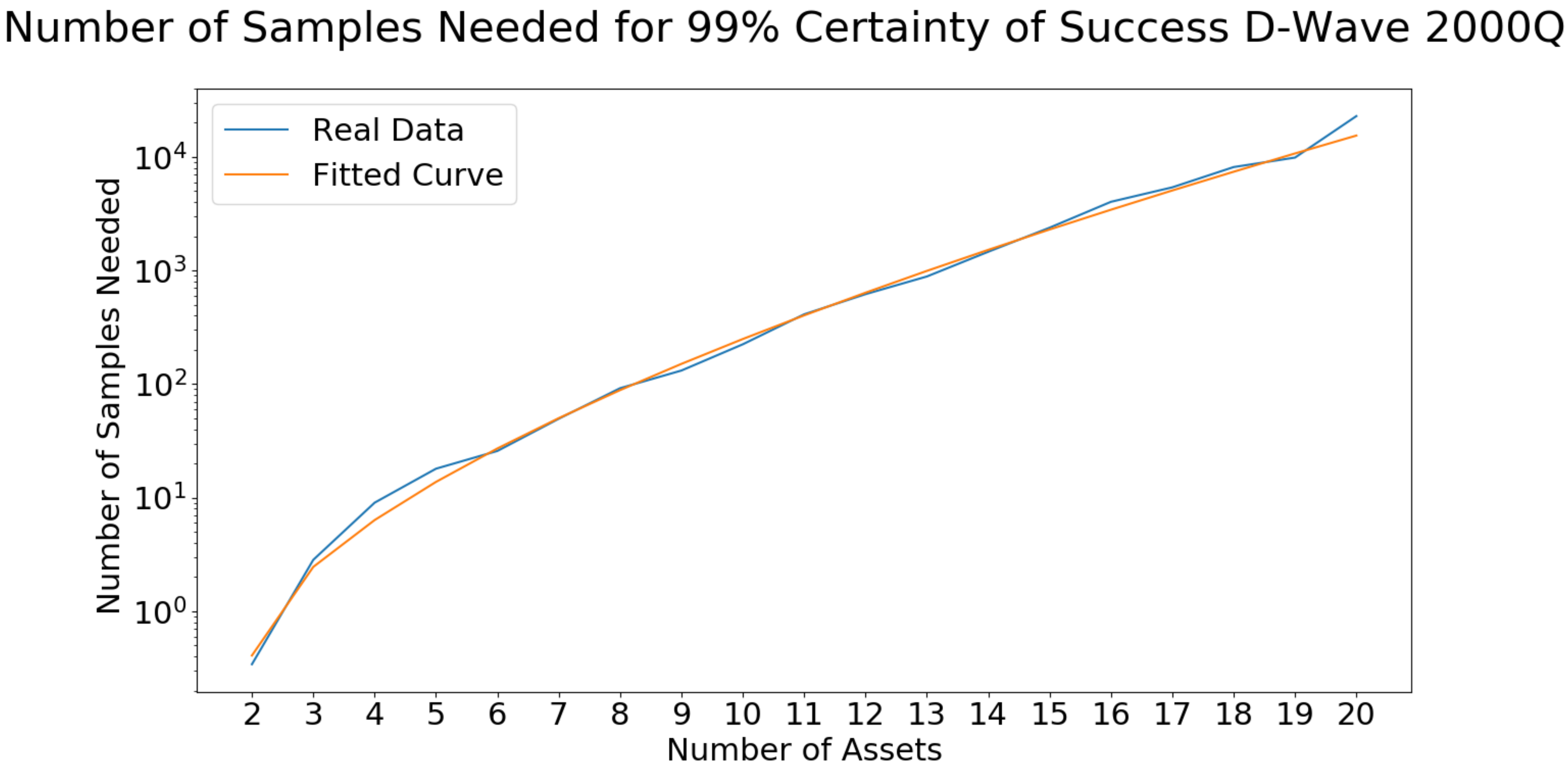
Number of Samples



$n \geq \frac{\log(1 - p_a)}{\log(1 - p_s)}$ where n is the number of samples, p_a is the desired accuracy, and p_s is probability of success

Number of Samples: Fitted to Log Scale



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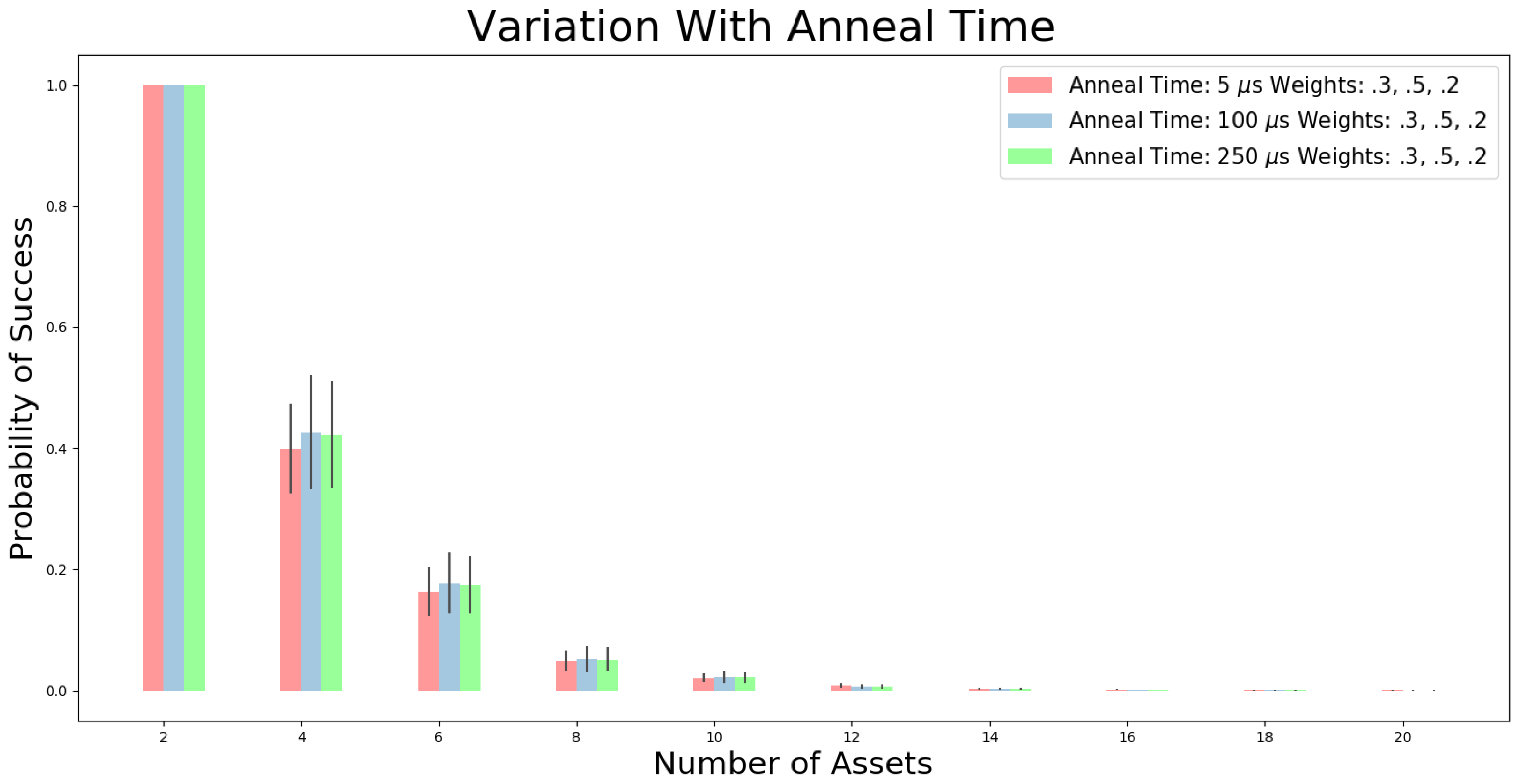
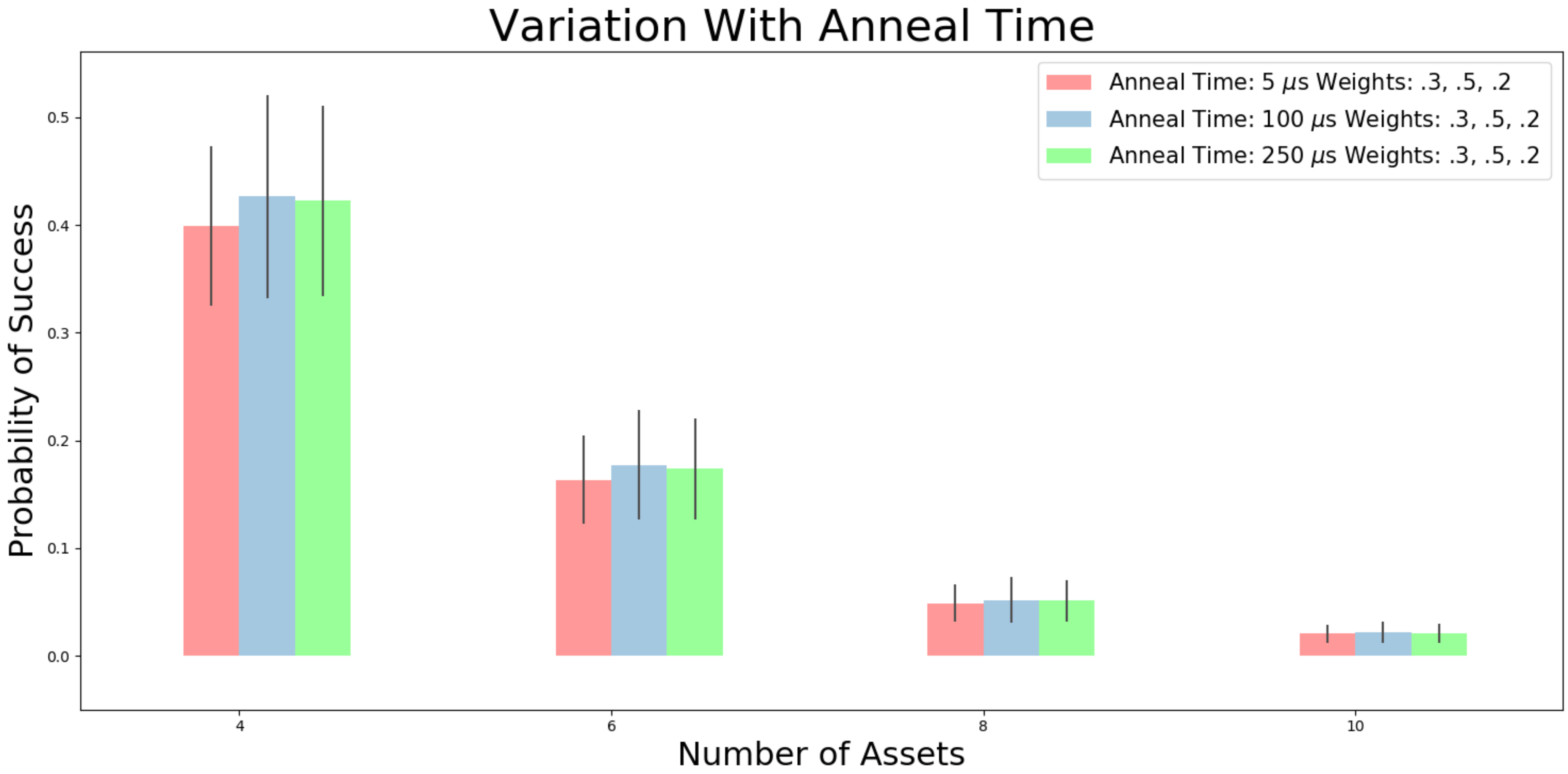
Probability of Success: Variation in Anneal Time



Probability of Success: Variation in Anneal Time



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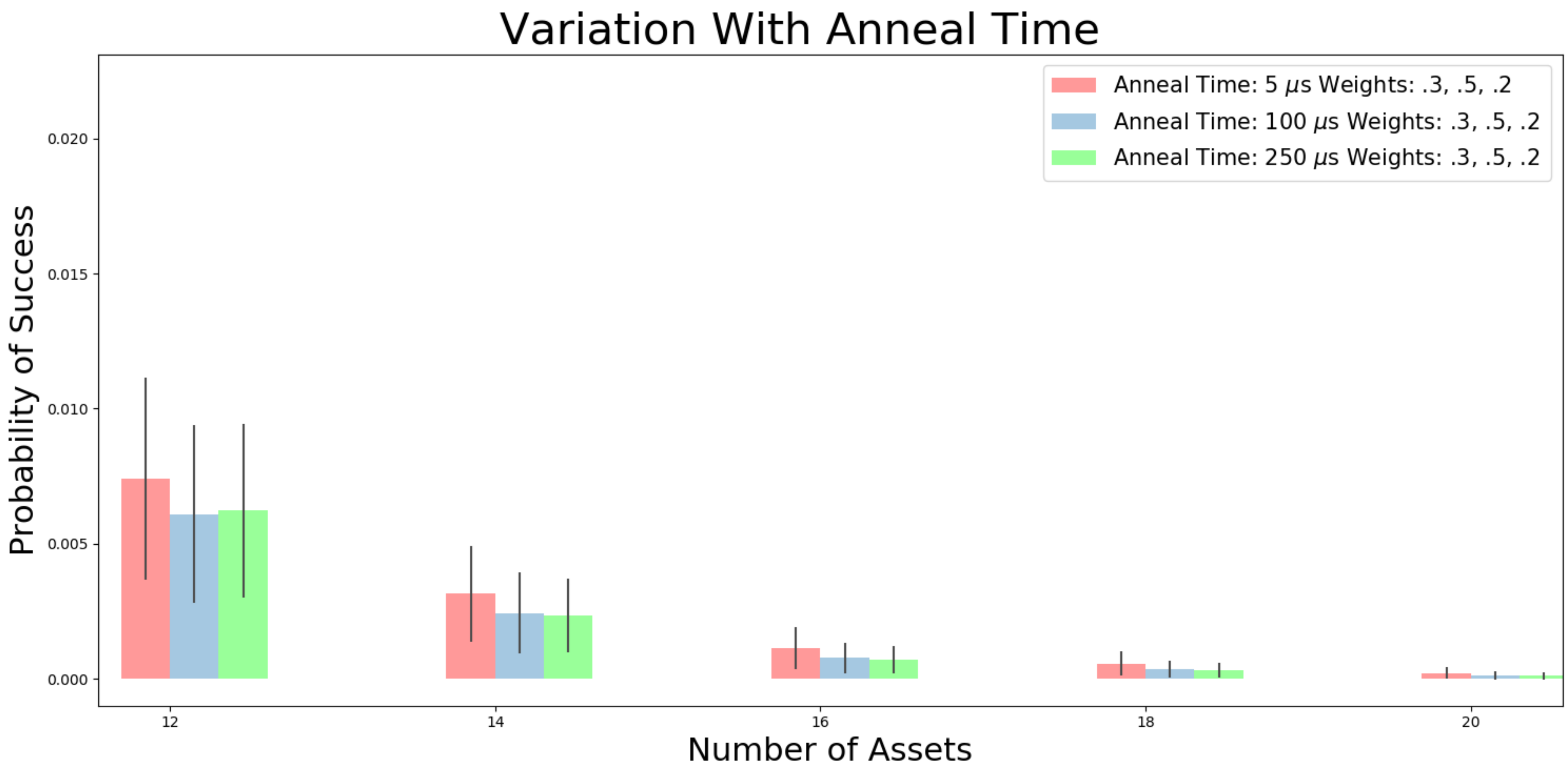
Probability of Success: Variation in Anneal Time



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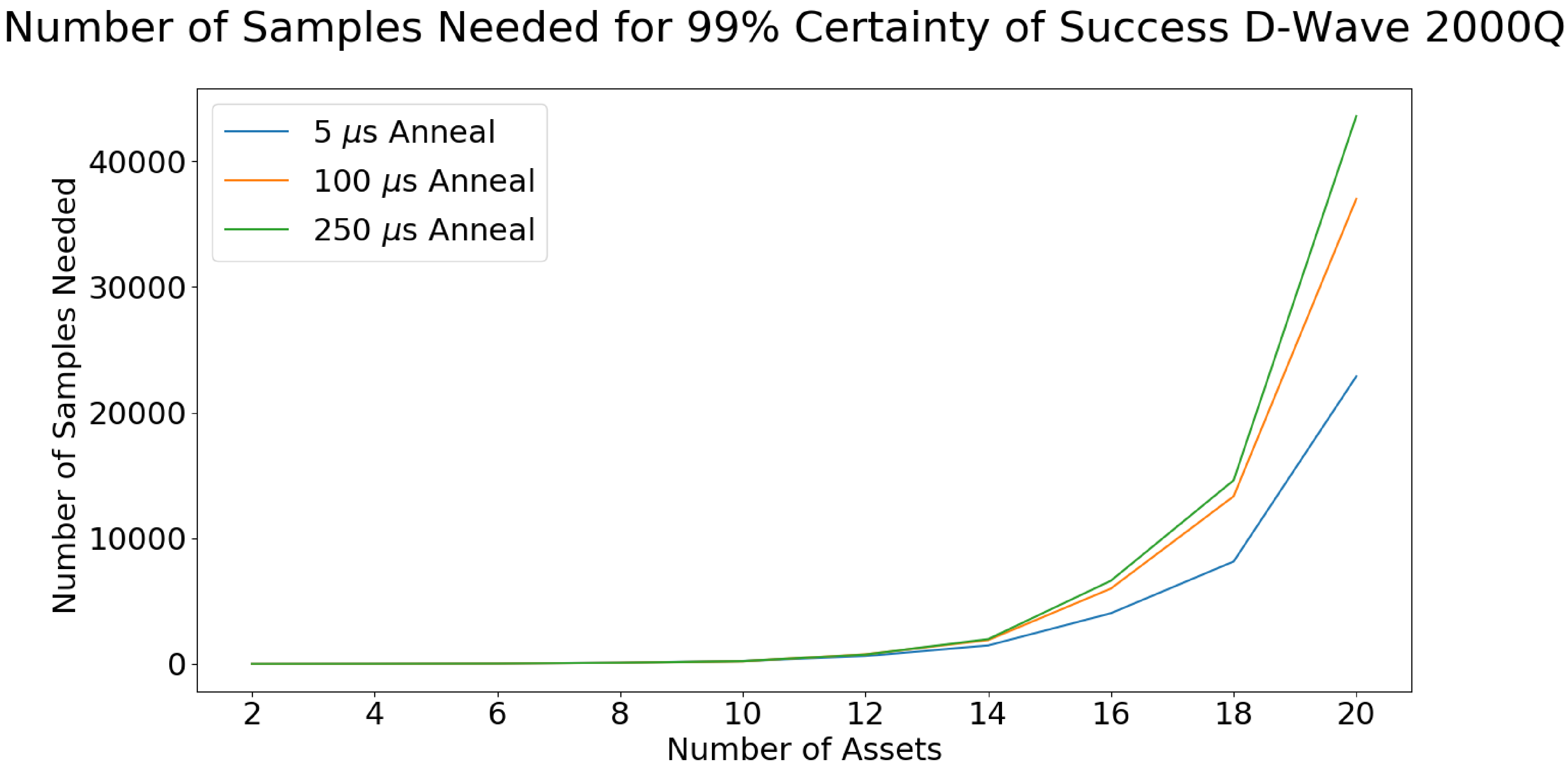
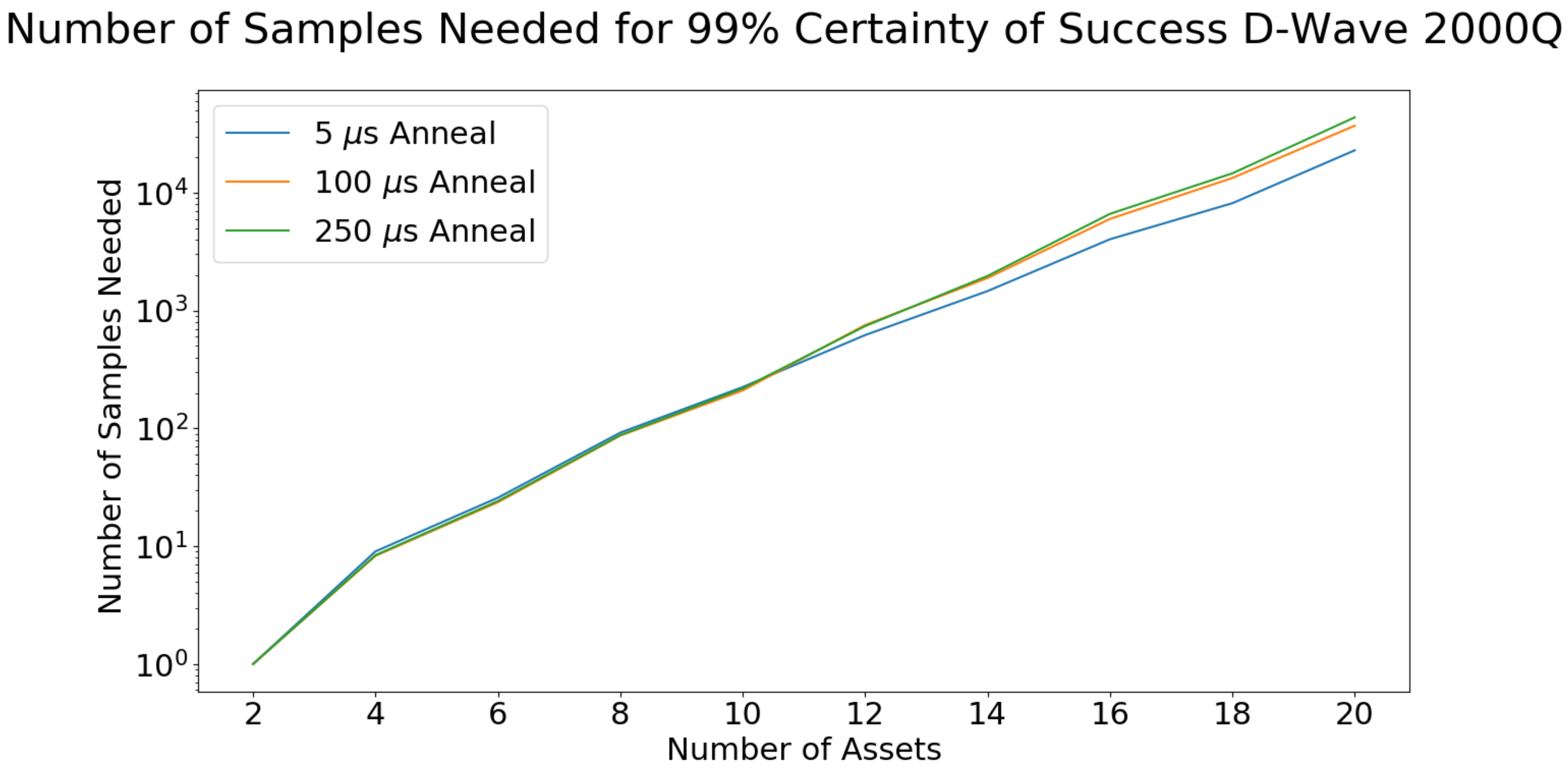
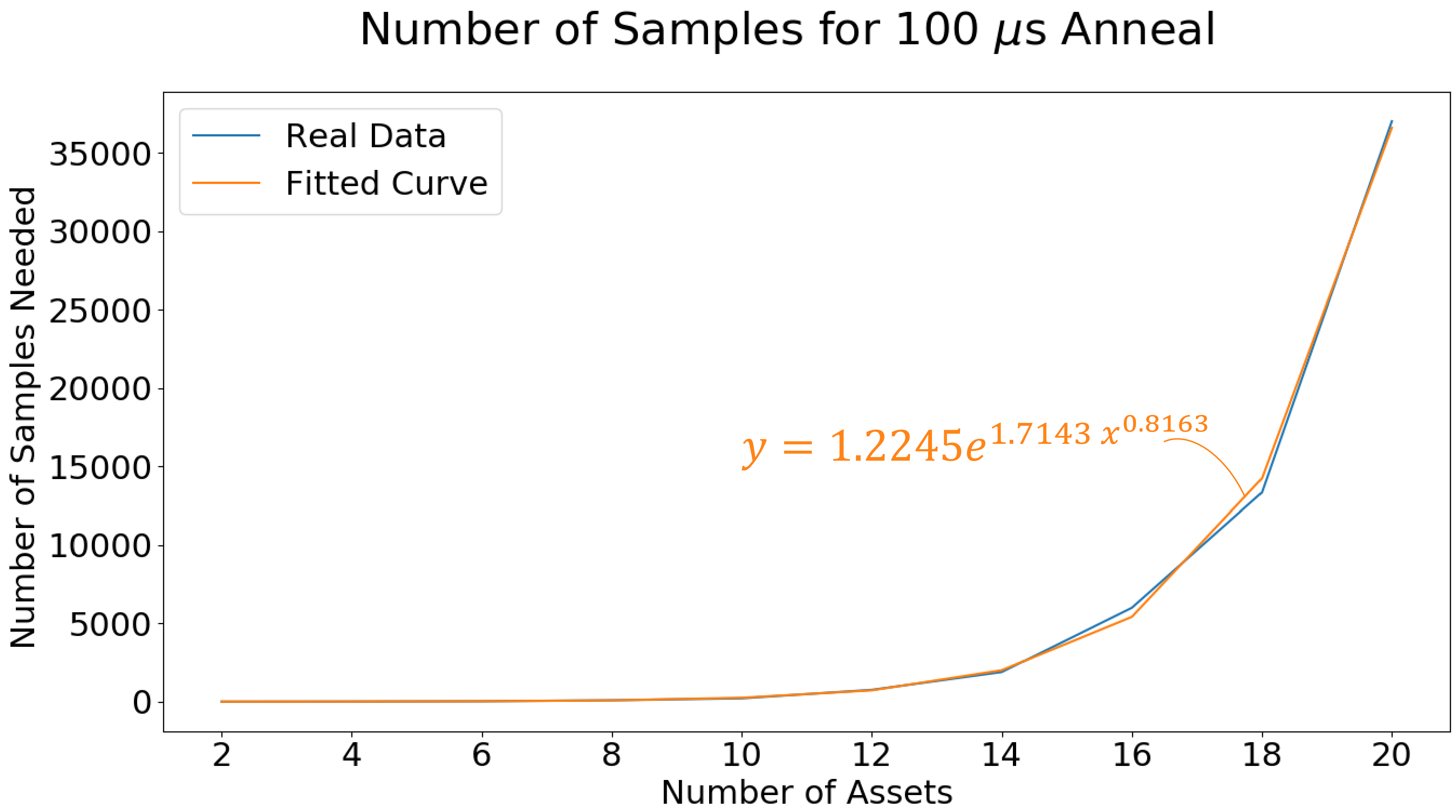
Number of Samples: Variation in Anneal Time



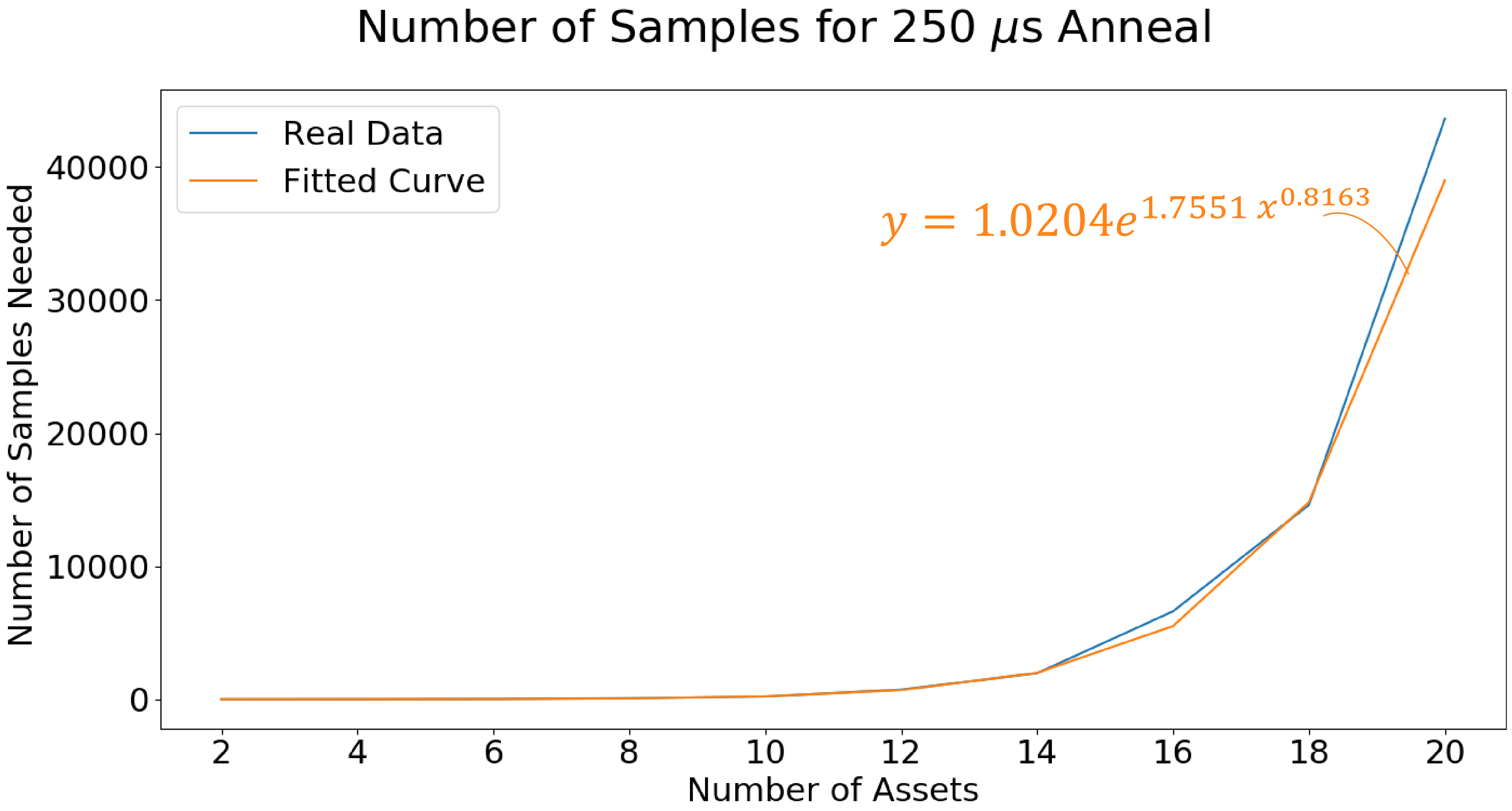
Number of Samples: Variation in Anneal Time



Number of Samples: 100μs Anneal



Number of Samples: 250μs Anneal



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Conclusions

- Lengthening the anneal time had small effects on the probability of success. For large problem size, small annealing time had a higher average while smaller problems yielded a higher average with longer anneal times.
- The number of samples needed to approach a 99% chance of finding the correct solution increased sub-exponentially with problem size through 20 assets.
- The weights on expected return, covariance, and budget penalty terms appear to have little to no effect on the probability of success.
- The time it takes to find an embedding on the D-Wave 2000Q increases sharply around 16 assets.

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Questions

University of Tennessee, Knoxville

Oak Ridge National Laboratory

Quantum Computing Institute

In collaboration with Khalifa University

Nada Elsokkary, Faisal Khan